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Phase diagram of the p-spin-interacting spin glass with ferromagnetic bias and a transverse field in the infinite-p limit

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The phase diagram of the p-spin-interacting spin glass model in a transverse field is investigated in the limit $p \to \infty$ under the presence of ferromagnetic bias. Using the replica method and the static approximation, we show that the phase diagram consists of four phases: Quantum paramagnetic, classical paramagnetic, ferromagnetic, and spin-glass phases. We also show that the static approximation is valid in the ferromagnetic phase in the limit $p \to \infty$ by using the large-p expansion. Since the same approximation is already known to be valid in other phases, we conclude that the obtained phase diagram is exact.

KEYWORDS: spin glass, p-spin interaction, random energy model, phase diagram, quantum effects

1. Introduction

A spin glass is a typical complex system characterized by quenched disorder and frustration. Numerous studies of spin glass systems clarified various interesting properties of disordered systems.[1,2] Incorporation of quantum effects into disordered systems has been of particular interest and studied intensively.[3–7] The noncommutativity of the operators makes the problem interesting but difficult, and a special care is required to obtain the correct result. The Trotter decomposition is known to be a useful approach to treat such effects.[8] In this approach, order parameters become dependent on the Trotter indices and are determined self-consistently.

Bray and Moore[3] proposed an approximate method to solve the problem. Their method, which is referred to as the static approximation (SA), is to neglect the time (or Trotter number) dependence of the order parameters. Using the SA, Thirumalai et al.[5] studied the Sherrington-Kirkpatrick (SK) model[9] in a transverse field to obtain the phase diagram. They pointed out the limitation of the SA, showing that the entropy does not vanish at zero temperature.

In order to understand the nature of quantum spin glasses, it would be helpful to investigate exactly solvable models. The infinite-range spin glass model with p-spin interactions is one of such tractable models. In the limit $p \to \infty$, this model is equivalent to the so-called Derrida's random energy model,[10] which can be exactly solvable in a simple way but retains nontrivial properties caused by quenched disorder. Goldschmidt[7] investigated this model in a

transverse field and obtained the phase diagram, which consists of a spin glass (SG) phase and two paramagnetic phases: One is the classical paramagnetic (CP) phase in which quantum fluctuations are irrelevant and the other is the quantum paramagnetic (QP) phase. He also suggested that the SA is exact in this model although there is no rigorous proof. Dobrosavljevic and Thirumalai[11] examined the validity of the SA in the same model by performing a systematic large-p expansion. While they showed that the phase diagram of Goldschmidt is correct, they also found that, for large but finite p, the SA is not valid in the QP phase.

This model is an extreme simplification of SG models but has a great advantage to be exactly solvable, which has enabled us to obtain many insights not only about SG properties[12] but also about the replica method itself[10] and information processing problems.[13] Our main aim in this paper is to investigate what happens in this model in a transverse field under the presence of ferromagnetic bias. The influence of quantum fluctuations in the ferromagnetic (F) phase is nontrivial, and it should be an interesting problem how the system behaves as a result of three competing effects of disorder, quantum fluctuations and ferromagnetic bias. In §2, we calculate the free energy of the model and give its phase diagram by using the replica method and the SA. In §3, we show the validity of the SA in the $p \to \infty$ limit by using systematic large-p expansion in the F phase. The last section is devoted to conclusion.

2. Replica analysis with the static approximation

2.1 Replica symmetric free energy

We consider the p-spin-interacting spin glass in a transverse field. Evaluation of the partition function can be carried out by a straightforward generalization of existing methods[5–7] to the case with ferromagnetic bias. The system is described by the following Hamiltonian:

$$H = -\sum_{i_1<\ldots<i_p} J_{i_1\ldots i_p}\sigma^z_{i_1}\ldots\sigma^z_{i_p} - \Gamma\sum_{i=1}^{N}\sigma^x_i \equiv T + V, \tag{1}$$

where i is the site index, σ^z and σ^x are Pauli spin operators, Γ denotes the strength of the transverse field and $J_{i_1\ldots i_p}$ is the quenched random interactions whose distribution function is given by

$$P(J_{i_1\ldots i_p}) = \left(\frac{N^{p-1}}{J^2\pi p!}\right)^{\frac{1}{2}} \exp\left\{-\frac{N^{p-1}}{J^2 p!}\left(J_{i_1\ldots i_p} - \frac{j_0 p!}{N^{p-1}}\right)^2\right\}. \tag{2}$$

The limit $p \to \infty$ is taken after all calculations. The partition function of this quantum system can be represented in terms of a corresponding classical spin system with the Ising variable $\sigma = (\pm 1)$ by the Trotter decomposition[8]

$$Z = \lim_{M\to\infty} \mathrm{Tr}\left(\mathrm{e}^{-\beta T/M}\mathrm{e}^{-\beta V/M}\right)^M = \lim_{M\to\infty} Z_M, \tag{3}$$

where

$$Z_M = C^{MN}\mathrm{Tr}\exp\left(\frac{\beta}{M}\sum_{t=1}^{M}\sum_{i_1<\ldots<i_p} J_{i_1\ldots i_p}\sigma_{i_1,t}\ldots\sigma_{i_p,t} + B\sum_{t=1}^{M}\sum_{i}\sigma_{i,t}\sigma_{i,t+1}\right), \tag{4}$$

and the constants B and C are related to the transverse field Γ as

$$B = \frac{1}{2}\ln\coth\frac{\beta\Gamma}{M}, \quad C = \left(\frac{1}{2}\sinh\frac{2\beta\Gamma}{M}\right)^{\frac{1}{2}}. \tag{5}$$

The symbol Tr denotes the trace over the σ-spins.

We use the replica method:[9]

$$[\log Z] = \lim_{n\to 0}\frac{[Z^n]-1}{n} \tag{6}$$

to carry out the random interactions $J_{i_1\dots i_p}$ averages $[\cdots]$. The replicated partition function is given by

$$[Z_M^n] = \mathrm{Tr}\exp\Bigg(\frac{\beta^2J^2N}{4M^2}\sum_{t,t'=1}^{M}\sum_{\mu,\nu=1}^{n}\left(\frac{1}{N}\sum_i\sigma_{i,t}^{\mu}\sigma_{i,t'}^{\nu}\right)^p + \frac{\beta j_0 N}{M}\sum_{t=1}^{M}\sum_{\mu=1}^{n}\left(\frac{1}{N}\sum_i\sigma_{i,t}^{\mu}\right)^p + B\sum_{t=1}^{M}\sum_{\mu=1}^{n}\sum_i\sigma_{i,t}^{\mu}\sigma_{i,t+1}^{\mu}\Bigg), \tag{7}$$

where the replica indices are denoted by μ and ν. We have omitted some irrelevant constants. The spin product term $\left(\sum_i\sigma_{i,t}^{\mu}\sigma_{i,t'}^{\nu}/N\right)^p$ can be simplified by introducing an order parameter $q_{tt'}^{\mu\nu}$ and its conjugate Lagrange multiplier $\widetilde{q}_{tt'}^{\mu\nu}$ for the constraint $q_{tt'}^{\mu\nu} = \sum_i\sigma_{i,t}^{\mu}\sigma_{i,t'}^{\nu}/N$. In the present case, we must distinguish diagonal $q_{tt'}^{\mu\mu}$ and off-diagonal $q_{tt'}^{\mu\nu}(\mu\neq\nu)$. Physically, $q_{tt'}^{\mu\mu}$ is a measure of quantum fluctuations and $q_{tt'}^{\mu\nu}$ is the SG order parameter. If there are no quantum fluctuations, the spin configuration $\sigma_{i,t}^{\mu}$ does not depend on time t and $q_{tt'}^{\mu\mu} = 1$. Quantum fluctuations reduce $q_{tt'}^{\mu\mu}$ from unity. Hence, $1 - q_{tt'}^{\mu\mu}$ gives a measure of quantum fluctuations. Also, to simplify $\left(\sum_i\sigma_{i,t}^{\mu}/N\right)^p$, the ferromagnetic order parameter m_t^{μ} is introduced and the constraint $m_t^{\mu} = \sum_i\sigma_{i,t}^{\mu}/N$ is imposed by the integration over the conjugate variable $\widetilde{m}_t^{\mu}$. Using these notations, we can rewrite the effective partition function as

$$\begin{aligned}[Z_M^n] = &\int\prod_{\mu}\prod_{t}dm_t^{\mu}d\widetilde{m}_t^{\mu}\prod_{\mu<\nu}\prod_{t,t'}dq_{tt'}^{\mu\nu}d\widetilde{q}_{tt'}^{\mu\nu}\prod_{\mu}\prod_{t\neq t'}dq_{tt'}^{\mu\mu}d\widetilde{q}_{tt'}^{\mu\mu}\\ &\times\exp N\Bigg\{\sum_{t,t'}\sum_{\mu<\nu}\left(\frac{\beta^2J^2}{2M^2}\left(q_{tt'}^{\mu\nu}\right)^p - \frac{1}{M^2}\widetilde{q}_{tt'}^{\mu\nu}q_{tt'}^{\mu\nu}\right) + \sum_{t,t'}\sum_{\mu}\left(\frac{\beta^2J^2}{4M^2}\left(q_{tt'}^{\mu\mu}\right)^p - \frac{1}{M^2}\widetilde{q}_{tt'}^{\mu\mu}q_{tt'}^{\mu\mu}\right)\\ &+\sum_t\sum_\mu\left(\frac{\beta^2 j_0}{M}\left(m_t^{\mu}\right)^p - \frac{1}{M}\widetilde{m}_t^{\mu}m_t^{\mu}\right) + \log\mathrm{Tr}\exp\left(-H_{\mathrm{eff}}\right)\Bigg\},\end{aligned} \tag{8}$$

where

$$H_{\mathrm{eff}} = -B\sum_t\sum_\mu\sigma_t^{\mu}\sigma_{t+1}^{\mu} - \frac{1}{M}\sum_\mu\sum_t\widetilde{m}_t^{\mu}\sigma_t^{\mu} - \frac{1}{M^2}\sum_{\mu<\nu}\sum_{t,t'}\widetilde{q}_{tt'}^{\mu\nu}\sigma_t^{\mu}\sigma_{t'}^{\nu} - \frac{1}{M^2}\sum_\mu\sum_{t\neq t'}\widetilde{q}_{tt'}^{\mu\mu}\sigma_t^{\mu}\sigma_{t'}^{\mu}. \tag{9}$$

We can calculate the free energy per spin F of the replicated systems in the thermodynamic

limit by the saddle-point method. The result is

$$-\beta F=\sum_{t,t'}\sum_{\mu<\nu}\left(\frac{\beta^2J^2}{2M^2}\left(q_{tt'}^{\mu\nu}\right)^p-\frac{1}{M^2}\widetilde{q}_{tt'}^{\mu\nu}q_{tt'}^{\mu\nu}\right)+\sum_{t,t'}\sum_{\mu}\left(\frac{\beta^2J^2}{4M^2}\left(q_{tt'}^{\mu\mu}\right)^p-\frac{1}{M^2}\widetilde{q}_{tt'}^{\mu\mu}q_{tt'}^{\mu\mu}\right)$$
$$+\sum_{t}\sum_{\mu}\left(\frac{\beta j_0}{M}\left(m_t^{\mu}\right)^p-\frac{1}{M}\widetilde{m}_t^{\mu}m_t^{\mu}\right)+\log\mathrm{Tr}\exp\left(-H_{\mathrm{eff}}\right), \quad (10)$$

where

$$q_{tt'}^{\mu\nu}=\langle\sigma_t^{\mu}\sigma_{t'}^{\nu}\rangle\ ,\ \widetilde{q}_{tt'}^{\mu\nu}=\frac{1}{2}\beta^2J^2p(q_{tt'}^{\mu\nu})^{p-1}, \quad (11a)$$

$$q_{tt'}^{\mu\mu}=\langle\sigma_t^{\mu}\sigma_{t'}^{\mu}\rangle\ ,\ \widetilde{q}_{tt'}^{\mu\mu}=\frac{1}{4}\beta^2J^2p(q_{tt'}^{\mu\mu})^{p-1}, \quad (11b)$$

$$m_t^{\mu}=\langle\sigma_t^{\mu}\rangle\ ,\ \widetilde{m}_t^{\mu}=\beta j_0p(m_t^{\mu})^{p-1}. \quad (11c)$$

The brackets $\langle\cdots\rangle$ denote the average by the weight $\exp(-H_{\mathrm{eff}})$. It is difficult to solve these equations exactly for arbitrary values of p because of the time dependence of the order parameters. However, in the limit $p\to\infty$, the problem is considerably simplified because conjugate variables can be either 0 or ∞. For example, if we restrict ourselves to the case that m_t^{μ} is non-negative (the other case can be treated similarly), eq. (11c) implies $0\le m_t^{\mu}\le 1$, which leads to either $\widetilde{m}_t^{\mu}\to 0$ $(0\le m_t^{\mu}<1)$ or $\widetilde{m}_t^{\mu}\to\pm\infty$ $(m_t^{\mu}\to 1)$. Therefore, the SA gives the exact phase boundaries as is shown below.

Now, we assume the replica symmetry (RS) and use the SA:

$$q_{tt'}^{\mu\mu}\equiv R\ ,\ q_{tt'}^{\mu\nu}\equiv q\ ,\ m_t^{\mu}\equiv m. \quad (12)$$

As already noted, R is the order parameter measuring the effect of quantum fluctuations, q is the conventional SG order parameter and m denotes the ferromagnetic order parameter. The free energy is then reduced to the expression

$$-\beta f\equiv-\lim_{M\to\infty}\lim_{n\to 0}\frac{\beta F}{n}$$
$$=\frac{1}{4}\beta^2J^2\left(R^p-q^p\right)+\frac{1}{2}\widetilde{q}q-\widetilde{R}R+\beta j_0m^p-\widetilde{m}m+\lim_{M\to\infty}\lim_{n\to 0}\frac{1}{n}\log\mathrm{Tr}\exp\left(-H_{\mathrm{eff}}\right). \quad (13)$$

Since all order parameters and the conjugate variables are independent of time and replica indices, the summation of spin products in H_{eff} is rewritten as

$$\sum_{\mu\ne\nu}\sum_{t,t'}\sigma_t^{\mu}\sigma_{t'}^{\nu}=\frac{1}{2}\left\{\left(\sum_{\mu}\sum_{t}\sigma_t^{\mu}\right)^2-\sum_{\mu}\left(\sum_{t}\sigma_t^{\mu}\right)^2\right\}. \quad (14)$$

Then, the effective Boltzmann factor $\exp(-H_{\mathrm{eff}})$ is considerably simplified by the Hubbard-Stratonovich transformation. We obtain

$$\exp(-H_{\mathrm{eff}})=\int Dz_1\prod_{\mu}\left\{\int Dz_2\exp\left(B\sum_{t}\sigma_t^{\mu}\sigma_{t+1}^{\mu}+\frac{A}{M}\sum_{t}\sigma_t^{\mu}\right)\right\}, \quad (15)$$

where

$$A=\sqrt{2\widetilde{R}-\widetilde{q}}\,z_2+\sqrt{\widetilde{q}}\,z_1+\widetilde{m}\ ,\ Dz=\frac{dz}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}z^2}. \quad (16)$$

Using the Trotter formula, we can take the spin trace in the limit $M \to \infty$ as

$$\lim_{M\to\infty} \mathrm{Tr}\exp\left(B\sum_t \sigma_t\sigma_{t+1} + \frac{A}{M}\sum_t \sigma_t\right) = \mathrm{Tr}e^{A\sigma^z+\beta\Gamma\sigma^x} = 2\cosh\sqrt{A^2+\beta^2\Gamma^2}. \tag{17}$$

Because each replica gives the same contribution to the replicated free energy, the limit of $n \to 0$ is easily taken. The result is

$$-\beta f = \frac{1}{4}\beta^2 J^2\left(R^p - q^p\right) + \frac{1}{2}\widetilde{q}q - \widetilde{R}R + \beta j_0 m^p - \widetilde{m}m + \int Dz_1\,\log\int Dz_2\,(2\cosh\omega), \tag{18}$$

where

$$\omega = \left(A^2+\beta^2\Gamma^2\right)^{\frac{1}{2}}. \tag{19}$$

The saddle-point conditions of the free energy are

$$\widetilde{m} = \beta j_0 p m^{p-1}, \tag{20}$$

$$\widetilde{R} = \frac{1}{4}\beta^2 J^2 p R^{p-1}, \tag{21}$$

$$\widetilde{q} = \frac{1}{2}\beta^2 J^2 p q^{p-1}, \tag{22}$$

$$m = \int Dz_1\,Y^{-1}\int Dz_2\,A\omega^{-1}\sinh\omega, \tag{23}$$

$$R = \int Dz_1\,Y^{-1}\left(\int Dz_2\,A^2\omega^{-2}\cosh\omega + \beta^2\Gamma^2\int Dz_2\,\omega^{-3}\sinh\omega\right), \tag{24}$$

$$q = \int Dz_1\,Y^{-2}\left(\int Dz_2\,A\omega^{-1}\sinh\omega\right)^2, \tag{25}$$

where

$$Y = \int Dz_2\,\cosh\omega. \tag{26}$$

When Γ is equal to 0, R becomes unity and the free energy is reduced to the classical result.[10,12] These eqs. (20)-(25) generalize the result of Goldschmidt[7] to the case of finite j_0.

2.2 Solutions of the equations of state.

The following inequality is useful to evaluate the solutions of eqs. (20)-(25).[14]

$$\begin{aligned} R &= \int Dz_1\,Y^{-1}\left(\int Dz_2\,A^2\omega^{-2}\cosh\omega + \beta^2\Gamma^2\int Dz_2\,\omega^{-3}\sinh\omega\right) \\ &\geq \int Dz_1\,Y^{-1}\int Dz_2\,A^2\omega^{-2}\cosh\omega \geq \int Dz_1\,Y^{-1}\int Dz_2\,A^2\omega^{-2}\sinh\omega \\ &\geq \int Dz_1\,Y^{-2}\left(\int Dz_2\,A\omega^{-1}\sinh\omega\right)^2 = q. \end{aligned} \tag{27}$$

From eqs. (20)-(22) and $0 \leq R, q, m \leq 1$, conjugate variables can be either 0 or ∞ in the limit $p \to \infty$. These conditions restrict combinations of the values of conjugate variables to

$$(\widetilde{m}, \widetilde{R}, \widetilde{q}) = (0,0,0), (0,\infty,0), (0,\infty,\infty), (\infty,0,0), (\infty,\infty,0), (\infty,\infty,\infty). \tag{28}$$

We must exclude the solution $(\widetilde{m}, \widetilde{R}, \widetilde{q}) = (\infty, 0, 0)$ as follows. Substituting these values $(\infty, 0, 0)$ into eqs. (23)-(25), we find the solution $(m, R, q) = (1, 1, 1)$, which contradicts the condition $(\widetilde{R}, \widetilde{q}) = (0, 0)$ as one can check from eqs. (21) and (22) with $p \to \infty$. Similarly, $(\widetilde{m}, \widetilde{R}, \widetilde{q}) = (\infty, \infty, 0)$ is also inappropriate. Finally, we get four solutions

$$(m, R, q) = (0, \frac{1}{\beta\Gamma} \tanh \beta\Gamma, 0), (0, 1, 0), (0, 1, 1), (1, 1, 1). \tag{29}$$

The solution $(m, R, q) = (0, 1, 0)$ is a paramagnetic one which is identical to that in the case without quantum fluctuations. In this solution quantum effects are irrelevant so that this phase is the CP phase. The solution $(m, R, q) = (0, \frac{1}{\beta\Gamma} \tanh \beta\Gamma, 0)$ represents a non-trivial paramagnetic phase which arises due to quantum fluctuations. R is reduced from unity by the transverse field Γ and this phase is the QP phase. The solution $(m, R, q) = (0, 1, 1)$ is for the SG phase. In the limit $p \to \infty$, the finite value solution of q is limited to $q = 1$ from eqs. (20)-(22). From the inequality $R \geq q$, R is also equal to 1 and hence we expect that quantum fluctuations are irrelevant in this phase.[11,15] The solution $(m, R, q) = (1, 1, 1)$ is ferromagnetic. In this phase, all order parameters are restricted to 1 in the limit $p \to \infty$. Therefore, quantum fluctuations are also irrelevant in this phase as will also be shown below. We summarize the above results in Table I.

Phase	(m, R, q)	$(\widetilde{m}, \widetilde{R}, \widetilde{q})$
CP	$(0, 1, 0)$	$(0, \infty, 0)$
QP	$(0, \tanh \beta\Gamma/(\beta\Gamma), 0)$	$(0, 0, 0)$
F	$(1, 1, 1)$	(∞, ∞, ∞)
SG	$(0, 1, 1)$	$(0, \infty, \infty)$

Table I. Values of order parameters in various phases.

Substituting the values of the order parameters and conjugate variables into eq. (18), we obtain the corresponding free energy as

$$f_{\mathrm{QP}} = -T \log 2 - T \log \cosh \beta\Gamma, \tag{30}$$

$$f_{\mathrm{CP}} = -\frac{1}{4}\beta J^2 - T \log 2, \tag{31}$$

$$f_{\mathrm{F}} = -j_0, \tag{32}$$

$$f_{\mathrm{SG}} \to -T\sqrt{\frac{2\widetilde{q}}{\pi}} \to -\infty \qquad (p \to \infty). \tag{33}$$

For this large p case the free energy of the SG phase is always smaller than the other free energies in eqs. (30), (31) and (32), although one knows that in spin glass phases maximization is the correct procedure.[1] However, this feature is an artifact of the RS solution.

2.3 Free energy in the spin glass phase

The correct solution of the SG phase is derived by the RSB. In the limit $p \to \infty$, the first step of the RSB (1RSB) is known to be sufficient in the classical case ($\Gamma = 0$).[12] We may expect that the same is true in the presence of a transverse field and therefore discuss the 1RSB scheme here. Note that at sufficiently low temperatures the full-step RSB is known to be necessary for finite p,[16,17] which we do not take into account explicitly here because we take the limit $p \to \infty$ in the end. New order parameters and a branch-point parameter m_1 are defined as follows

$$q^{l\mu_l,l\nu_l} = q_0 \ , \ q^{l\mu_l,l'\nu_{l'}} = q_1 \qquad l \neq l', \tag{34}$$

where $l = 1, \ldots, n/m_1$ is the block number and $\mu_l = 1, \ldots, m_1$ is the index inside a block. Detailed calculations are given in Appendix A. The 1RSB free energy is found to be given as

$$\begin{aligned}\beta f =& \frac{1}{2}m_1 \left(\frac{1}{2}\beta^2 J^2 (q_0)^p - \widetilde{q}_0 q_0\right) + \frac{1}{2}(1-m_1)\left(\frac{1}{2}\beta^2 J^2 (q_1)^p - \widetilde{q}_1 q_1\right) \\ &- \frac{1}{4}\beta^2 J^2 (R)^p + \widetilde{R}R - \beta j_0 m^p + \widetilde{m}m - \frac{1}{m_1}\int Dz_1 \log \int Dz_2 \left(\int Dz_3\, 2\cosh\omega_1\right)^{m_1}\end{aligned} \tag{35}$$

where

$$\omega_1 = \left(A_1^2 + \beta^2\Gamma^2\right)^{\frac{1}{2}} \ , \ A_1 = \sqrt{\widetilde{q}_0}\, z_1 + \sqrt{\widetilde{q}_1 - \widetilde{q}_0}\, z_2 + \sqrt{2\widetilde{R} - \widetilde{q}_1}\, z_3 \ + \widetilde{m}. \tag{36}$$

If we set $q_0 = q_1$, the RS solution (18) is reproduced. The saddle-point conditions of the 1RSB free energy are given by

$$\widetilde{m} = \beta j_0 p m^{p-1}, \tag{37}$$

$$\widetilde{R} = \frac{1}{4}\beta^2 J^2 p R^{p-1}, \tag{38}$$

$$\widetilde{q}_0 = \frac{1}{2}\beta^2 J^2 p q_0^{p-1}, \tag{39}$$

$$\widetilde{q}_1 = \frac{1}{2}\beta^2 J^2 p q_1^{p-1}, \tag{40}$$

$$m = \int Dz_1\, Y_1^{-1} \int Dz_2\, Y_2^{m_1-1}, \int Dz_3\, A_1\omega_1^{-1} \sinh\omega_1, \tag{41}$$

$$R = \int Dz_1\, Y_1^{-1} \int Dz_2\, Y_2^{m_1-1} \left(\int Dz_3\, A_1^2\omega_1^{-2}\cosh\omega_1 + \beta^2\Gamma^2 \int Dz_3\, \omega_1^{-3}\sinh\omega_1\right), \tag{42}$$

$$q_0 = \int Dz_1 \left(Y_1^{-1}\int Dz_2\, Y_2^{m_1-1}\int Dz_3\, A_1\omega_1^{-1}\sinh\omega_1\right)^2, \tag{43}$$

$$q_1 = \int Dz_1\, Y_1^{-1}\int Dz_2\, Y_2^{m_1-2}\left(\int Dz_3\, A_1\omega_1^{-1}\sinh\omega_1\right)^2, \tag{44}$$

where

$$Y_1 = Y_1(z_1) \equiv \int Dz_2 \left(\int Dz_3\, \cosh\omega_1\right)^{m_1} \ , \ Y_2 = Y_2(z_1, z_2) \equiv \int Dz_3\, \cosh\omega_1. \tag{45}$$

Inequalities $R \geq q_1 \geq q_0$ are also derived in a similar way to the derivation of eq. (27) as explained in detail in Appendix A. In the limit $p \to \infty$, we find from these inequalities

and eqs. (37)-(44) that the only non-trivial RSB solution is $(m, R, q_0, q_1) = (0, 1, 0, 1)$. The corresponding free energy is

$$f = -\frac{1}{4}\beta J^2 m_1 - T\frac{1}{m_1}\log 2. \tag{46}$$

Taking a variation with respect to m_1, we find

$$f_{\rm SG} = -J\sqrt{\log 2}. \tag{47}$$

This is the correct free energy of the SG phase.

We can determine all the phase boundaries by equating free energies between different phases. The phase diagram thus obtained is shown in Figs. 1-4. We can see that, as Γ grows, quantum fluctuations reduce the ferromagnetic order and cause a phase transition to the QP phase as in Fig. 2. Order parameters discontinuously change at any phase boundary. In that sense, all the phase transitions are of first order.

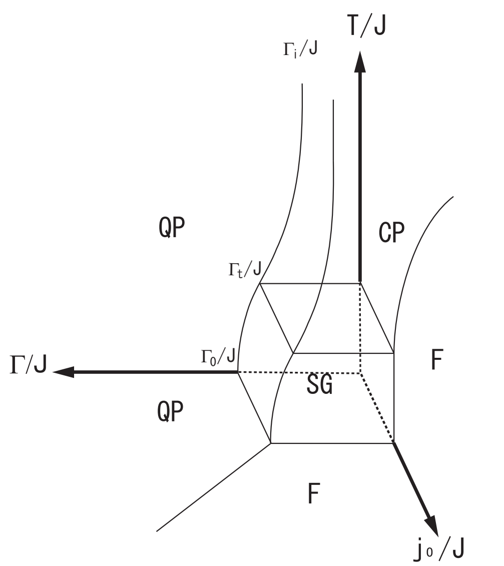

Fig. 1. Full phase diagram of the model in the limit $p \to \infty$. The QP phase appears when Γ/J becomes larger than $\Gamma_i/J = 1/\sqrt{2}$. The CP phase is completely suppressed by quantum fluctuations in the range of $\Gamma/J \geq \Gamma_t/J = \log(2+\sqrt{3})/(2\sqrt{2})$. Replica-symmetry breaking exists only in the SG phase.

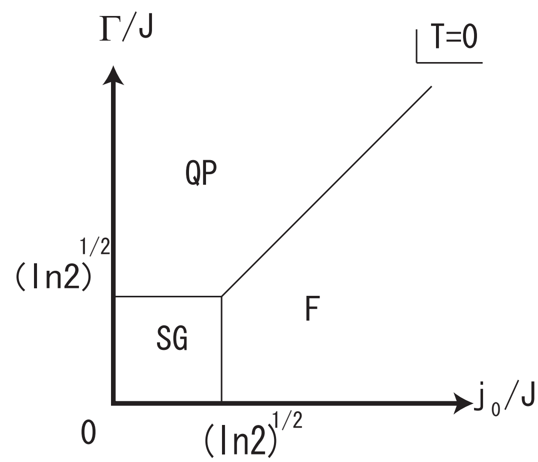

Fig. 2. Ground-state phase diagram on the $T = 0$ plane of Fig. 1. For large Γ, quantum fluctuations destroy the ferromagnetic order and cause a phase transition to the QP phase.

3. Validity of the static ansatz

3.1 Expansion from the large-p limit

In this section, we check the validity of the SA. Again, the method is generalization of ref. 11 to the case with ferromagnetic bias. For that purpose, we introduce corrections to the

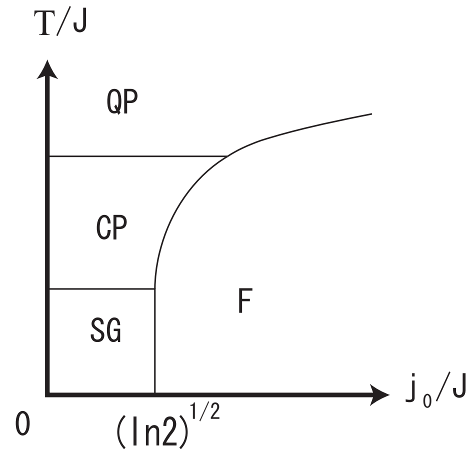

Fig. 3. Schematic phase diagram on the T-j_0 plane in the range of $\Gamma_i \leq \Gamma \leq \Gamma_t$. For large T, the QP phase appears instead of the CP phase. As Γ grows, the CP phase diminishes and vanishes at $\Gamma = \Gamma_t$.

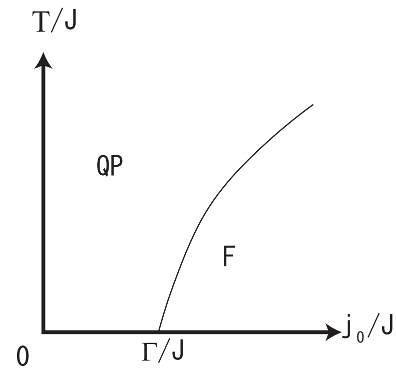

Fig. 4. Phase diagram on the T-j_0 plane for large Γ. The SG phase disappears at $\Gamma/J = \Gamma_0/j = \sqrt{\log 2}$.

SA with t, t'-dependence and expand the free energy with respect to those correction terms. Then, it is shown that the time-dependent parts are irrelevant in the limit $p \to \infty$.

We start from the RS solution

$$q^{\mu\mu}_{t,t'} = R(t,t') \ , \ q^{\mu\nu}_{t,t'} = q(t,t') \ , \ m^{\mu}_{t} = m(t). \tag{48}$$

Separating each conjugate variable to the static and time-dependent parts, we rewrite the effective Hamiltonian $H_{\rm eff} = H_{\rm stat} + V(t,t')$ as

$$H_{\rm stat} = -B\sum_{\mu}\sum_{t}\sigma^{\mu}_{t}\sigma^{\mu}_{t+1} - \frac{1}{M^2}\widetilde{R}\sum_{\mu}\sum_{t\neq t'}\sigma^{\mu}_{t}\sigma^{\mu}_{t'} - \frac{1}{2M^2}\widetilde{q}\sum_{\mu\neq\nu}\sum_{t,t'}\sigma^{\mu}_{t}\sigma^{\nu}_{t'} - \frac{1}{M}\widetilde{m}\sum_{\mu}\sum_{t}\sigma^{\mu}_{t}, \tag{49}$$

$$V(t,t') = -\frac{1}{M^2}\sum_{\mu}\sum_{t\neq t'}\Delta\widetilde{R}(t,t')\sigma^{\mu}_{t}\sigma^{\mu}_{t'} - \frac{1}{2M^2}\sum_{\mu\neq\nu}\sum_{t,t'}\Delta\widetilde{q}(t,t')\sigma^{\mu}_{t}\sigma^{\nu}_{t'} - \frac{1}{M}\sum_{\mu}\sum_{t}\Delta\widetilde{m}(t)\sigma^{\mu}_{t}, \tag{50}$$

where

$$\widetilde{R}(t,t') = \widetilde{R} + \Delta\widetilde{R}(t,t') \ , \ \widetilde{q}(t,t') = \widetilde{q} + \Delta\widetilde{q}(t,t') \ , \ \widetilde{m}(t,t') = \widetilde{m} + \Delta\widetilde{m}(t,t'). \tag{51}$$

It is expected that the order parameters are monotone decreasing functions of the time interval $|t-t'|$ because they are originally written as spin-correlation functions eqs. (11a)-(11c). Then, the time-dependent parts of their conjugate variables, which are the pth powers of the order parameters, become drastically small for large p. Therefore, it is reasonable to expand the free energy with respect to the time-dependent part. The free energy is expanded with respect to

$V(t,t')$ to first order as

$$\beta f = \frac{1}{2}\frac{1}{M^2}\sum_{t,t'}\left(\frac{1}{2}\beta^2 J^2 q(t,t')^p - \widetilde{q}q(t,t') - \Delta\widetilde{q}(t,t')q(t,t')\right)$$
$$-\frac{1}{M^2}\sum_{t\neq t'}\left(\frac{1}{4}\beta^2 J^2 R(t,t')^p - \widetilde{R}R(t,t') - \Delta\widetilde{R}(t,t')R(t,t')\right)$$
$$-\frac{1}{M}\sum_{t}\left(\beta j_0 m(t)^p - \widetilde{m}m(t) - \Delta\widetilde{m}(t)m(t)\right) - \lim_{n\to 0}\frac{1}{n}\left(\log \mathrm{Tr}\exp(-H_{\mathrm{stat}}) + \langle V\rangle_{\mathrm{stat}}\right). \quad (52)$$

The brackets $\langle\cdots\rangle_{\mathrm{stat}}$ denote the average by the weight $\exp(-H_{\mathrm{stat}})$. The equation satisfied by each order parameter can be obtained by taking a functional derivative with respect to the time-dependent part of the conjugate variable. The results are

$$m(t) = \lim_{n\to 0}\frac{1}{n}\left\langle\sum_{\mu}\sigma_t^{\mu}\right\rangle_{\mathrm{stat}}, \quad (53a)$$

$$q(t,t') = -\lim_{n\to 0}\frac{1}{n}\left\langle\sum_{\mu\neq\nu}\sigma_t^{\mu}\sigma_{t'}^{\nu}\right\rangle_{\mathrm{stat}}, \quad (53b)$$

$$R(t,t') = \lim_{n\to 0}\frac{1}{n}\left\langle\sum_{\mu}\sigma_t^{\mu}\sigma_{t'}^{\mu}\right\rangle_{\mathrm{stat}}. \quad (53c)$$

The new Boltzmann factor $\exp(-H_{\mathrm{stat}})$ is identical to $\exp(-H_{\mathrm{eff}})$ under the SA and can be simplified by the Hubbard-Stratonovich transformation as eq. (15). From eqs. (15) and (53a)-(53c), we can verify that q and m are time-independent and only R is time-dependent because of the independence of each replica in H_{stat} and the translational invariance in the Trotter direction. Then, the problem is reduced to the evaluation of the correlation function of the one-dimensional Ising system in a field. Calculations are somewhat involved but straightforward. Details are given in Appendix B. The result is

$$R(\tau) = \int Dz_1\, Y^{-1}\left(\int Dz_2\, A^2\omega^{-2}\cosh\omega + \beta^2\Gamma^2\int Dz_2\,\omega^{-2}\cosh\omega(1-2\tau)\right). \quad (54)$$

We have used the same notation as in eqs. (16)-(19) and the continuous-time notation

$$\tau \equiv \lim_{M\to\infty}\frac{t-t'}{M}. \quad (55)$$

The result (24) of the SA is reproduced by integrating eq. (54) over τ. The equations of other order parameters are identical to the result under the SA. Hence, from eq. (27), we see that the inequality $R(t,t') \geq q$ still holds for any τ.

3.2 Explicit solution of $R(\tau)$ in the ferromagnetic phase

In the F phase, all conjugate variables go to infinity in the limit $p\to\infty$ and it is reasonable to assume $2\widetilde{R} = \widetilde{q}$ according to eqs. (21) and (22). Hence, the integration over z_2 just gives 1 and we find

$$R_{\mathrm{F}}(\tau) = \int Dz_1\left(A^2\omega^{-2} + \beta^2\Gamma^2\omega^{-2}\frac{\cosh\omega(1-2\tau)}{\cosh\omega}\right). \quad (56)$$

The conjugate variables $\widetilde{m}$ and $\widetilde{q}$ are proportional to p and very large which enables us to derive the leading finite-p correction by the systematic large-p expansion.[11,15] For large conjugate parameters, we can estimate the integral (56) by the saddle-point method for fixed τ. To compare the time-dependent result with that under the SA, we start from the result of the SA. Under the condition $2\widetilde{R} = \widetilde{q}$, eq. (24) reads

$$R = \int Dz_1 \left(A^2\omega^{-2} + \beta^2\Gamma^2 \int Dz_2\, \omega^{-3} \frac{\sinh\omega}{\cosh\omega} \right). \tag{57}$$

The first term on the right-hand side of eq. (57) is rewritten as

$$\int Dz_1\, A^2\omega^{-2} = \int Dz_1 \frac{\left(1 + \sqrt{\widetilde{q}}z_1/\widetilde{m}\right)^2}{1 + 2\sqrt{\widetilde{q}}z_1/\widetilde{m} + \left(\sqrt{\widetilde{q}}z_1/\widetilde{m}\right)^2 + (\beta\Gamma/\widetilde{m})^2}. \tag{58}$$

For large p, because $1/\widetilde{m} \propto 1/p$ is very small, we can expand the right-hand side of eq. (58). After straightforward calculations, we obtain the leading $1/p$-correction term as

$$\int Dz_1\, A^2\omega^{-2} \approx 1 - \frac{\beta^2\Gamma^2}{\widetilde{m}^2} = 1 - \left(\frac{\Gamma}{j_0}\right)^2 \frac{1}{p^2} m^{-2p+2}. \tag{59}$$

Similarly, the second term on the right-hand side of eq. (57) is rewritten as

$$\int Dz_1 \left(\beta^2\Gamma^2\omega^{-3} - 2\beta^2\Gamma^2\omega^{-3}\frac{\mathrm{e}^{-2\omega}}{1+\mathrm{e}^{-2\omega}} \right), \tag{60}$$

and $\exp(-2\omega)$ is exponentially small for large p. The first term of eq. (60) can be evaluated by the series expansion with respect to $1/p$ as in eq. (59). The result is

$$\int Dz_1\, \beta^2\Gamma^2\omega^{-3} \approx \frac{\Gamma^2}{\beta j_0^3}\frac{1}{p^3} m^{-3p+3}. \tag{61}$$

This term is at most proportional to $1/p^3$ and is negligible. We can also evaluate other order parameters m and q. Under the condition $2\widetilde{R} = \widetilde{q}$, eq. (23) is rewritten as

$$\begin{aligned} m &= \int Dz_1\, Y^{-1} \int Dz_2\, A\omega^{-1}\sinh\omega = \int Dz_1\, A\omega^{-1}\tanh\omega \\ &\approx \int Dz_1\, A\omega^{-1} \approx 1 - \frac{1}{2}\left(\frac{\Gamma}{j_0}\right)^2 \frac{1}{p^2} m^{-2p+2} \approx 1 - \frac{1}{2}\left(\frac{\Gamma}{j_0}\right)^2 \frac{1}{p^2}. \end{aligned} \tag{62}$$

Similarly, eq. (25) reads

$$\begin{aligned} q &= \int Dz_1\, Y^{-2} \left(\int Dz_2\, A\omega^{-1}\sinh\omega \right)^2 = \int Dz_1\, \left(A\omega\tanh\omega\right)^2 \\ &\approx \int Dz_1\, A^2\omega^{-2} \approx 1 - \left(\frac{\Gamma}{j_0}\right)^2 \frac{1}{p^2} = R. \end{aligned} \tag{63}$$

This equation is consistent with the condition $2\widetilde{R} = \widetilde{q}$.

Next, we proceed to a time-dependent analysis. From eq. (56), its first term on the right-hand side gives the same corrections as in the SA case. We rewrite the second term as

$$\int Dz_1\, \omega^{-2}\frac{\cosh\omega(1-2\tau)}{\cosh\omega} = \int Dz_1\, \omega^{-2}\frac{\mathrm{e}^{-2\tau\omega} + \mathrm{e}^{-\omega(2-2\tau)}}{1+\mathrm{e}^{-2\omega}} \approx \int Dz_1\, \omega^{-2}\left(\mathrm{e}^{-2\tau\omega} + \mathrm{e}^{-\omega(2-2\tau)}\right). \tag{64}$$

This integration can be evaluated by the saddle-point method. The saddle-point condition of $\omega^{-2}\exp(-2\tau\omega)$ is

$$z^2 = 4\tau^2\widetilde{q}\left(1 - \frac{\beta^2\Gamma^2}{\omega^2}\right). \tag{65}$$

It is difficult to solve this equation exactly. However, for large p, the second term on the right-hand side of eq. (65) is vanishingly small and we can approximate the saddle point as $z = \pm 2\tau\sqrt{\widetilde{q}}$. The contribution from the saddle point $z = 2\tau\sqrt{\widetilde{q}}$ is

$$\begin{aligned}&\left\{(2\tau\widetilde{q} + \widetilde{m})^2 + \beta^2\Gamma^2\right\}^{-1}\exp\left(-2\tau^2\widetilde{q} - 2\tau\sqrt{(2\tau\widetilde{q} + \widetilde{m})^2 + \beta^2\Gamma^2}\right)\\ &\approx \frac{1}{\beta^2 j_0^2}\frac{1}{p^2}\exp\left\{-\left(\tau^2 J^2\beta^2 + 2\tau\sqrt{(\tau^2 J^2\beta^2 + \beta j_0) + \frac{\beta^2\Gamma^2}{p^2}}\right)p\right\}. \end{aligned}\tag{66}$$

The other saddle point $z = -2\tau\sqrt{\widetilde{q}}$ gives a similar contribution. Replacing 2τ by $2 - 2\tau$, we can also obtain the saddle-point value of $\omega^{-2}\exp(-\omega(2-2\tau))$. Then, the explicit result of $R(\tau)$ in the F phase is given by

$$R_{\mathrm{F}}(\tau) \approx 1 - \left(\frac{\Gamma}{j_0}\right)^2\frac{1}{p^2} + \left(\frac{\Gamma}{j_0}\right)^2\frac{1}{p^2}f(\tau, p) \tag{67}$$

where the function $f(\tau, p)$ expresses the time-dependent correction which decreases exponentially as p grows. Because the third term of eq. (67) is vanishingly small, the main finite-p correction is the second term, which is identical to the SA result. Accordingly, in the F phase, the time-dependent part of the finite-p correction is exponentially small for large p as in the CP and SG phases[11,15] and the SA is valid in that sense. We also calculated the free energy to the order $1/p$. However, we found that the first order correction vanishes and the free energy remains as in eq. (32). To obtain the leading order in $1/p$, we must proceed to the next order approximation but it is beyond our purpose in this paper.

3.3 Remarks

In the F phase, we have found that the leading corrections of order parameters are actually time independent. For other phases, previous works revealed that in the CP and SG phases similar results hold.[11,15] Hence, in these instances, the SA is valid not only in the limit $p \to \infty$ but also as long as p is adequately large. Meanwhile, in the QP phase, strong disagreement occurs between the SA and the time-dependent analysis for large but finite p.[11] Not only the spin autocorrelation function $R(\tau)$ is actually time dependent, but also the low temperature behaviour shows violations of the thermodynamic law within the SA. From these facts, we may conclude that the SA well describes the region in which quantum fluctuations are weak, but strong quantum effects lead to a collapse of this approximation and unphysical behaviours of thermodynamic quantities. In spite of such inexpediency, the free energy recovers the SA results in the limit $p \to \infty$ even in the QP phase. Consequently, the SA gives correct free energies in all the phases in the limit $p \to \infty$, and the phase diagram depicted in Figs. 1-4 should be exact.

4. Conclusion

In this paper, we have studied the p-spin-interacting spin glass model with ferromagnetic bias in a transverse field by the replica method. Trotter decomposition has been employed to reduce the quantum system to a classical one and the SA has been assumed to obtain the solutions of the equations of state. We have clarified the structure of the full phase diagram, which consists of four phases, the CP, QP, F, and SG phases.

We have also checked the validity of the SA in the F phase by the large-p expansion. Leading finite-p corrections of the order parameters have been calculated and it has been shown that they are actually time-independent. It is known that similar results hold in the CP and SG phases. This is not the case for the QP phase. Nevertheless the free energy of the QP phase turns out to be identical to the SA result in the limit $p \to \infty$. In that sense, the SA gives correct solutions in this limit and our phase diagram is exact.

Admittedly, the model investigated in this paper is not a faithful reproduction of real SG systems. However, it has a great advantage that order parameters and the free energy can be exactly obtained. Goldschmidt found two qualitatively different types of paramagnetic phases, CP and QP. We also found that quantum fluctuations reduce the ferromagnetic order and cause a transition to the QP phase. These properties appear to be plausible in more realistic systems. On the other hand, we saw that in the CP, F, and SG phases quantum fluctuations are completely irrelevant, which should be specific to this model. For more realistic models (like the SK model) we should take into account the influence of quantum fluctuations on order parameters. However, it is difficult to treat such an effect as was done in the present paper and we need different techniques. Finding effective approaches and improving the SA remain interesting problems to be investigated in the future. Our present results will serve as a first step to understanding the interplay between quantum fluctuations, ferromagnetic bias and quenched disorder.

After submission of the manuscript, we came to notice that the same problem was discussed by Saakyan[18] and Jun-Ichi Inoue.[19] The originality of our work lies in the systematic analysis of the validity of the SA and the explicit clarification of the structure of the full phase diagram.

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Appendix A: 1RSB free energy and the inequalities $R \geq q_1 \geq q_0$

In this Appendix we derive the 1RSB free energy eq. (35) and the inequalities $R \geq q_1 \geq q_0$. Under the 1RSB scheme and the SA, eq. (10) reads

$$\beta f = \frac{1}{2}m_1\left(\frac{1}{2}\beta^2 J^2 q_0^p - \widetilde{q}_0 q_0\right) - \frac{1}{2}(m_1-1)\left(\frac{1}{2}\beta^2 J^2 q_1^p - \widetilde{q}_1 q_1\right) - \left(\frac{1}{4}\beta^2 J^2 R^p - \widetilde{R}R\right)$$
$$- (\beta j_0 m^p - \widetilde{m}m) - \lim_{M\to\infty}\lim_{n\to 0}\frac{1}{n}\log \mathrm{Tr}\exp\left(-H_{\mathrm{eff}}\right). \quad \text{(A·1)}$$

The effective Hamiltonian H_{eff} is written in the form

$$-H_{\mathrm{eff}} = B\sum_{\mu}\sum_{t}\sigma_t^{\mu}\sigma_{t+1}^{\mu} + \widetilde{m}\sum_{\mu}\sum_{t}\sigma_t^{\mu} + \widetilde{R}\sum_{\mu}\sum_{t\neq t'}\sigma_t^{\mu}\sigma_{t'}^{\mu}$$
$$+\frac{1}{2}\left(\widetilde{q}_0\sum_{\mu\neq\nu}\sum_{t,t'}\sigma_t^{\mu}\sigma_{t'}^{\nu} + (\widetilde{q}_1-\widetilde{q}_0)\sum_{l}^{n/m_1}\sum_{\mu_l\neq\nu_l}^{\text{block}}\sum_{t,t'}\sigma_t^{\mu_l}\sigma_{t'}^{\nu_l}\right). \quad \text{(A·2)}$$

We can rewrite the summation of spin products as

$$-H_{\mathrm{eff}} = \frac{1}{2}\widetilde{q}_0\left(\sum_{\mu}\sum_{t}\sigma_t^{\mu}\right)^2 + \frac{1}{2}\left(\widetilde{q}_1-\widetilde{q}_0\right)\sum_{l}^{n/m_1}\left\{\left(\sum_{\mu_l}\sum_{t}\sigma_t^{\mu_l}\right)^2 - \sum_{\mu_l}\left(\sum_{t}\sigma_t^{\mu_l}\right)^2\right\}$$
$$+\left(\widetilde{R}-\frac{1}{2}\widetilde{q}_0\right)\sum_{\mu}\left(\sum_{t}\sigma_t^{\mu}\right)^2 + B\sum_{\mu}\sum_{t}\sigma_t^{\mu}\sigma_{t+1}^{\mu} + \widetilde{m}\sum_{\mu}\sum_{t}\sigma_t^{\mu}. \quad \text{(A·3)}$$

To take the spin trace, the Hubbard-Stratonovich transformation is employed for the quadratic terms. The result is

$$\mathrm{e}^{-H_{\mathrm{eff}}} = \int Dz_1 \prod_{l}^{n/m_1}\left\{\int Dz_2 \prod_{\mu_l=(l-1)m_1+1}^{lm_1}\left(\int Dz_3\,\mathrm{e}^{L}\right)\right\}, \quad \text{(A·4)}$$

where $L \equiv A_1/M\sum_t \sigma_t^{\mu_l} + B\sum_t \sigma_t^{\mu_l}\sigma_{t+1}^{\mu_l}$ and A_1 is defined in eq. (36). Using the Trotter formula, we can take the limit $M\to\infty$ and perform the spin trace as in eq. (17). The result in the limit $n\to 0$ is given as

$$\lim_{M\to\infty}\lim_{n\to 0}\frac{1}{n}\log\mathrm{Tr}\exp\left(-H_{\mathrm{eff}}\right) = \frac{1}{m_1}\int Dz_1\,\log\int Dz_2\left(\int Dz_3\,2\cosh\omega_1\right)^{m_1}. \quad \text{(A·5)}$$

The equations of state eqs. (37)-(44) are obtained by taking a variation of the free energy eq. (35) with respect to conjugate variables and order parameters.

Next, we derive the inequalities $R \geq q_1 \geq q_0$. From eqs. (41)-(44),

$$\begin{aligned} R \;=\; & \int Dz_1\,Y_1^{-1}\int Dz_2\,Y_2^{m_1-1}\left(\int Dz_3\,A_1^2\omega_1^{-2}\cosh\omega_1 + \beta^2\Gamma^2\int Dz_3\,\omega_1^{-3}\sinh\omega_1\right) \\ & \geq \int Dz_1\,Y_1^{-1}\int Dz_2\,Y_2^{m_1-1}\int Dz_3\,A_1^2\omega_1^{-2}\cosh\omega_1 \\ & \geq \int Dz_1\,Y_1^{-1}\int Dz_2\,Y_2^{m_1-1}\int Dz_3\,A_1^2\omega_1^{-2}\sinh\omega_1 \\ & \geq \int Dz_1\,Y_1^{-1}\int Dz_2\,Y_2^{m_1-2}\left(\int Dz_3\,A_1\omega_1^{-1}\sinh\omega_1\right)^2 = q_1. \end{aligned} \quad \text{(A·6)}$$

Similarly, we can show $q_1 \geq q_0$ from the definition of Y_1 (45) as

$$\begin{aligned} q_1 &= \int Dz_1\, Y_1^{-1} \int Dz_2\, Y_2^{m_1-2} \left(\int Dz_3\, A_1 \omega_1^{-1} \sinh \omega_1 \right)^2 \\ &\geq \int Dz_1 \left(Y_1^{-1} \int Dz_2\, Y_2^{m_1-1} \int Dz_3\, A_1 \omega_1^{-1} \sinh \omega_1 \right)^2 = q_0. \end{aligned} \tag{A·7}$$

Appendix B: Evaluations of the correlation function

We calculate the correlation function and derive the expression for $R(\tau)$ eq. (54). The unnormalized correlation function of the one-dimensional Ising system with periodic boundary is

$$G(t,t') = \mathrm{Tr}\sigma_t \sigma_{t'} \exp\left(J \sum_{t=1}^{M} \sigma_t \sigma_{t+1} + h \sum_{t=1}^{M} \sigma_t \right). \tag{B·1}$$

We can compute the correlation function $G(t,t')$ by the transfer matrix method. The general solution is

$$G(t,t') = 4x_+^2 x_-^2 \left(\lambda_+^{t-t'} \lambda_-^{M-(t-t')} + \lambda_+^{M-(t-t')} \lambda_-^{t-t'} \right) + (2x_+^2 - 1)^2 \lambda_+^M + (2x_-^2 - 1)^2 \lambda_-^M, \tag{B·2}$$

where $\lambda_\pm$ are the eigenvalues of the transfer matrix and $x_\pm$ are the first components of the eigenvectors $|\pm\rangle$. Their explicit forms are

$$\lambda_\pm = \mathrm{e}^J \left(\cosh h \pm \sqrt{\cosh^2 h - 1 + \mathrm{e}^{-4J}} \right), \tag{B·3}$$

$$|\pm\rangle = D_\pm \begin{pmatrix} -\mathrm{e}^{-J} \\ \mathrm{e}^J \left(\sinh h \mp \sqrt{\sinh^2 h + \mathrm{e}^{-4J}} \right) \end{pmatrix} \equiv \begin{pmatrix} x_\pm \\ y_\pm \end{pmatrix}, \tag{B·4}$$

where $D_\pm$ are normalization constants. In the present case, the parameters $J = B = \log(\coth \beta\Gamma/M)^{1/2}$ and $h = A/M$ depend on the number of spins M. Restoring the omitted overall factor $C = \{(1/2)\sinh 2\beta\Gamma/M\}^{1/2}$, we can obtain the finite value of the correlation function in the limit $M \to \infty$. After straightforward calculations, we get

$$x_\pm^2 \to \frac{1}{2} \frac{(\beta\Gamma)^2}{\omega^2 \mp A\omega}\ ,\ (C\lambda_\pm)^M \to \mathrm{e}^{\pm\omega}, \tag{B·5}$$

where ω is given in eq. (19). Hence, the correlation function $G(t,t')$ is given by

$$G(t,t') \to G(\tau) = 2A^2 \omega^{-2} \cosh \omega + 2(\beta\Gamma)^2 \omega^{-2} \cosh \omega(1 - 2\tau). \tag{B·6}$$

Substituting eq. (B·6) into eq. (53c) and taking the limit $n \to 0$, we finally get

$$R(\tau) = \int Dz_1\, Y^{-1} \left(\int Dz_2\, A^2 \omega^{-2} \cosh \omega + \beta^2 \Gamma^2 \int Dz_2\, \omega^{-2} \cosh \omega(1 - 2\tau) \right).$$

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